Nonlinear two-dimensional frequency- and temperature-dependent vortex dynamics in a tilted washboard pinning potential

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Abstract. The Langevin equation for a two-dimensional nonlinear guided vortex motion in a tilted cosine pinning potential in the presence of an *ac* current is *exactly* solved in terms of a *matrix* continued fraction at arbitrary value of the Hall effect. The influence of an *ac* current of arbitrary amplitude and frequency on the *dc* and *ac* magnetoresistivity tensors is theoretically considered. Experimental realization of this model in thin-film geometry opens up a possibility for a variety of experimental studies of directed motion of vortices under ($dc + ac$)- driving simply by measuring longitudinal and transverse voltages.

1. Introduction

As the pinning force in a washboard planar pinning potential (WPPP) is directed perpendicular to the washboard channels, the vortices generally tend to move along these channels. Such a *guided* motion of vortices in the presence of the Hall effect produces anisotropic transport behaviour for which even (+) and odd (−) (with respect to the magnetic field reversal) longitudinal ($\|$) and transverse ($\perp$) *dc* nonlinear magnetoresistivities $\rho^{\pm}_{\|,\perp}$ depend nontrivially on the angle α between the *dc* current density vector $\mathbf{j}$ and the direction of the WPPP channels ("guiding direction"). The *dc*-current nonlinear guiding problem recently has been extensively studied both theoretically [1-3] and experimentally [4-6]; it was exactly solved for the washboard WPPP within the framework of the two-dimensional ($2D$) single-vortex stochastic model of anisotropic pinning based on the Fokker-Planck equation and rather simple formulas were derived for the *dc* magnetoresistivities $\rho^{\pm}_{\|,\perp}$ [2,3].

On the other hand, in contrast to the *dc* current case, the temperature-dependent *ac*-driven vortex motion problem *in theory* has been exactly solved only for one-dimensional ($1D$) *nontilted* cosine pinning potential at a small oscillation amplitude of the vortices [7-8]. At the same time it is evident that the examination of a strong nonlinear *ac*-driven response of the vortices in the presence of arbitrary *dc* bias will be interesting both for theory and for different high-frequency or microwave applications.

In the following we study $2D$ vortex dynamics in the presence of ($dc + ac$)-driven Lorentz forces of arbitrary magnitude. For the exact solution of this problem we used the matrix continued fraction technique earlier suggested and later extensively employed for calculation of $1D$ nonlinear ($ac + dc$)-driven response of overdamped Josephson junction with noise in [9,10].

As a result, two groups of new findings were obtained. First, for the previously solved in [1,2] $2D$ dc-problem the influence of ac current on the overall shape and appearance of the Shapiro steps on the anisotropic dc $\rho^{\pm}_{\parallel,\perp}$- CVC's was calculated and analyzed. Second, for the ac current at a frequency ω plus dc bias the $2D$ nonlinear time-dependent stationary $\rho^{ac\pm}_{\parallel,\perp}$ ac-response on the frequency ω in terms of nonlinear impedance tensor $\hat{Z}$ and a nonlinear ac response at ω-harmonics was studied. Below we present a short derivation of main analytical results that allows to consider these findings in the future publications.

2. Formulation of the problem

The Langevin equation for a vortex moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B} = \mathbf{n}B$ ($B \equiv |\mathbf{B}|$, $\mathbf{n} = n\mathbf{z}$, $\mathbf{z}$ is the unit vector in the z direction and $n = \pm 1$) has the form

$$\eta \mathbf{v} + n\alpha_H \mathbf{v} \times \mathbf{z} = \mathbf{F}_L + \mathbf{F}_p + \mathbf{F}_{th}, \tag{1}$$

where $\mathbf{F}_L = n(\Phi_0/c)\mathbf{j} \times \mathbf{z}$ is the Lorentz force (Φ_0 is the magnetic flux quantum, c is the speed of light), $\mathbf{j} = \mathbf{j}(t) = \mathbf{j}^{dc} + \mathbf{j}^{ac}\cos\omega t$, where $\mathbf{j}^{dc}$ and $\mathbf{j}^{ac}$ are the dc and ac current density amplitudes and ω is the angular frequency, $\mathbf{F}_{th}$ is the thermal fluctuation force represented by a Gaussian white noise with zero mean [1,2,7,8], η is the vortex viscosity, α_H is the Hall constant, $\mathbf{F}_p = -\nabla U_p(x)$ is the anisotropic pinning force and we assume, as usual [1,7] an a-periodic pinning potential of the form $U_p(x) = (U_p/2)(1 - \cos kx)$ where $k = 2\pi/a$.

The main quantity of physical interest in our problem is the average electric field, induced by the moving vortex system, which is given by

$$\langle \mathbf{E} \rangle = (n/c)\mathbf{B} \times \langle \mathbf{v} \rangle = n(B/c)(-\langle v_y \rangle \mathbf{x} + \langle v_x \rangle \mathbf{y}), \tag{2}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the unit vectors in the x and y directions, respectively. As follows from Eq. (1) $\langle v_y \rangle = F_{Ly}/\eta + \delta \langle v_x \rangle$ where $\delta \equiv n\epsilon$, $\epsilon \equiv \alpha_H/\eta$, and so for determination of $\langle \mathbf{E} \rangle$ from Eq. (2) it is sufficient to calculate the $\langle v_x \rangle$ from Eq. (1). This calculation gives

$$\langle v_x \rangle(t) = (\Phi_0 j_c / c\eta D)[j^{dc} + j^{ac}\cos\omega t - \langle \sin \mathrm{x} \rangle(t)], \tag{3}$$

where $D \equiv 1 + \delta^2$, $j^{dc} \equiv n(j^{dc}_y + \delta j^{dc}_x)/j_c$, $j^{ac} \equiv n(j^{ac}_y + \delta j^{ac}_x)/j_c$, $j_c \equiv cF_p/\Phi_0$, $j_y = j\cos\alpha$, $j_x = j\sin\alpha$, $\langle \sin \mathrm{x} \rangle(t) = (i/2)[\langle r \rangle(t) - \langle r^{-1} \rangle(t)]$, $r(t) = exp[-i\mathrm{x}(t)]$, and $\mathrm{x} = kx$.

Looking for only the stationary ac response, which is independent of the initial condition, one may seek all the $\langle r^m \rangle(t)$ in the form [9]

$$\langle r^m \rangle(t) = \sum_{k=-\infty}^{\infty} F^m_k(\omega) e^{ik\omega t}. \tag{4}$$

On substituting Eq. (4) into Eq. (3) we obtain recurrence equations for the $(j^{ac}, j^{dc}, \Omega, g)$-dependent Fourier amplitudes F^m_k, where $g = (U_p/2T)$, T is the temperature in energy units, $\Omega = \omega\hat{\tau}$ is the dimensionless frequency and $\hat{\tau} \equiv \eta D/kF_p$ is the relaxation time. From the solution of these recurrence equations in terms of matrix continued fraction (see details in Ref. [9]) we can find the dimensionless average pinning force $\langle F_{px} \rangle(t)$ which is the main anisotropic nonlinear component of our theory

$$\langle \hat{F}_{px} \rangle(t) = -\langle \sin \mathrm{x} \rangle(t) = \sum_{k=0}^{\infty} \mathrm{Im}(\psi_k e^{ik\omega t}), \tag{5}$$

where $\psi_0 \equiv F^1_0(\omega)$ and for $k \geq 1$ we have $\psi_k \equiv F^1_k(\omega) - F^{-1}_k(\omega)$. From Eq. (5) we can decompose average pinning force $\langle \hat{F}_{px} \rangle(t)$ into three components

$$\langle \hat{F}_{px} \rangle(t) = \langle \hat{F}_{px} \rangle^{\omega}_0 + \langle \hat{F}_{px} \rangle_{t1} + \langle \hat{F}_{px} \rangle^{k>1}_t. \tag{6}$$

In Eq. (6) $\langle \hat{F}_{px}\rangle_0^\omega \equiv -\langle \sin \mathrm{x}\rangle_0^\omega = \mathrm{Im}\psi_0$ is the time independent (but frequency dependent) *static* average pinning force; $\langle \hat{F}_{px}\rangle_{t1} \equiv -\langle \sin \mathrm{x}\rangle_{t1} = \mathrm{Im}(\psi_1 e^{i\omega t})$ is the time-dependent *dynamic* average pinning force; $\langle \hat{F}_{px}\rangle_t^{k>1} \equiv -\langle \sin \mathrm{x}\rangle_t^{k>1} = \mathrm{Im}(\psi_k e^{ik\omega t})$ describes a contribution of the harmonics with $k > 1$ into the dynamic average pinning force.

3. Ω-dependent nonlinear dc resistivity response

In order to derive the ω-dependent nonlinear *dc* resistivity and conductivity tensors we first express (see Eq. (2)) the time independent part of $\langle E_y\rangle(t)$ and $\langle E_x\rangle(t)$ as

$$\begin{aligned} \langle E_y\rangle_0^\omega &= (\rho_f/D)\nu_0^\omega(j_y^{dc} + \delta j_x^{dc}), \\ \langle E_x\rangle_0^\omega &= (\rho_f/D)[j_x^{dc}(1+\delta^2(1-\nu_0^\omega)) - \delta j_y^{dc}]. \end{aligned} \tag{7}$$

Here $\rho_f \equiv B\Phi_0/\eta c^2$ is the flux-flow resistivity, $\nu_0^\omega \equiv 1 - \langle \sin \mathrm{x}\rangle_0^\omega / j^{dc} = 1 + \langle \hat{F}_{px}\rangle_0^\omega / j_{dc}$, where ν_0^ω at $j^{ac} = 0$ can be considered as the probability of vortex hopping over the pinning potential barrier under the influence of the dimensionless generalized moving force $\hat{F}_{Lx}^{dc} = j^{dc}$ in the x direction [2].

From Eqs. (7) we find the magnetoresistivity tensor for the *dc* nonlinear law $\langle \mathbf{E}_0^\omega(\omega)\rangle = \hat{\rho}_0^\omega \mathbf{j}^{dc}$ as

$$\hat{\rho}_0^\omega = \begin{pmatrix} \rho_{xx}^{dc} & \rho_{xy}^{dc} \\ \rho_{yx}^{dc} & \rho_{yy}^{dc} \end{pmatrix} = \frac{\rho_f}{D}\begin{pmatrix} D - \delta^2\nu_0^\omega & -\delta\nu_0^\omega \\ \delta\nu_0^\omega & \nu_0^\omega \end{pmatrix}. \tag{8}$$

The experimentally observable longitudinal and transverse (with respect to the $\mathbf{j}$ direction) *dc* magnetoresistivities $\rho_\parallel = E_\parallel/j$ and $\rho_\perp = E_\perp/j$ (where j is the *dc* current density $j^2 = j_x^2 + j_y^2$) have the form

$$\begin{aligned} \rho_\parallel &= (\rho_f/D)[(D - \delta^2\nu_0^\omega)\sin^2\alpha \ + \nu_0^\omega \cos^2\alpha], \\ \rho_\perp &= (\rho_f/D)[\delta\nu_0^\omega - D(1-\nu_0^\omega)\cos\alpha\sin\alpha], \end{aligned} \tag{9}$$

where in order to separate the even and odd components of $\rho_{\parallel,\perp}$ we should use $\nu_0^{\omega\pm}(n) = [\nu_0^\omega(n) \pm \nu_0^\omega(-n)]/2$ which are the even and odd components (relative to the magnetic field inversion) of the function $\nu_0^\omega(n)$ (compare with Eqs. (13)-(14) in Ref. [3]).

4. Nonlinear stationary ac response

Using Eq. (2) we determine nonlinear stationary *ac* response as

$$\langle \mathbf{E}\rangle_t \equiv \langle \langle \mathbf{E}\rangle(t) - \langle \mathbf{E}\rangle_0^\omega \rangle = (nB/c)[\langle v_x\rangle_t \mathbf{y} - \langle v_y\rangle_t \mathbf{x}], \tag{10}$$

where $\langle \mathbf{E}\rangle_0^\omega$ is the time-independent part of $\langle \mathbf{E}\rangle(t)$ (see also Eqs. (7)), whereas $\langle v_y\rangle_t$ and $\langle v_x\rangle_t$ are time-dependent periodic parts of $\langle v_y\rangle(t)$ and $\langle v_x\rangle(t)$. From Eqs. (10) we have

$$\langle E_y\rangle_t = (n\rho_f j_c/D)\sum_{k=1}^{\infty}(j^{ac})^k \mathrm{Re}\{Z_k(\omega)e^{ik\omega t}\}, \tag{11}$$

where $Z_k(\omega) = \delta_{1,k} - i\psi_k(\omega)/(j^{ac})^k$ and $\delta_{1,k}$ is Kronecker's delta. The dimensionless transformation coefficients Z_k in Eq. (11) have a physical meaning of the k-th harmonic with frequency $\Omega_k \equiv k\omega$ in the *ac* nonlinear $\langle E_y\rangle_t$ response. $Z_k(\omega)$ for $k = 1$ yields the dimensionless nonlinear impedance Z_1 as

$$Z_1 = 1 - i\psi_1/j^{ac} = \rho_1 - i\zeta_1 \tag{12}$$

where ρ_1 and ζ_1 are the dynamic resistivity and the reactivity, respectively. Now using Eqs. (10) and (11) we can express the nonlinear stationary *ac* responses E^{ac}_{y1} and E^{ac}_{x1} on the frequency ω in terms of Z_1 as

$$\begin{gathered} E^{ac}_{y1} = (\rho_f/D)(\delta j^{ac}_x + j^{ac}_y)\mathrm{Re}\{Z_1 e^{i\omega t}\}, \\ E^{ac}_{x1} = (\rho_f/D)\mathrm{Re}\{e^{i\omega t}[(D-\delta^2 Z_1)j^{ac}_x - \delta Z_1 j^{ac}_y)]\}. \end{gathered} \tag{13}$$

From Eqs. (13) follows that the complex amplitudes of the electric field $\boldsymbol{E}_1$ and the current density $\boldsymbol{J} = \mathbf{j}^{ac}e^{i\omega t}$ are connected by the relation $\boldsymbol{E}_1 = \hat{Z}\boldsymbol{J}$, where $\hat{Z}$ is the frequency- dependent nonlinear impedance tensor

$$\hat{Z}(\omega) = \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix} = \frac{\rho_f}{D}\begin{pmatrix} D-\delta^2 Z_1 & -\delta Z_1 \\ \delta Z_1 & Z_1 \end{pmatrix}. \tag{14}$$

It is relevant to remark the similarity of Eq. (8) and Eq. (14) from which follows that for the *ac* response Z_1 plays the same role as ν^{ω}_0 for the *dc* response. It is easy to show that $\langle \hat{F}_{px}\rangle_{t1} = \mathrm{Re}\{e^{i\omega t}[j^{ac}(Z_1 - 1)]\}$ which gives a physical interpretation of the Z_1 impedance.

The real quantities $\mathbf{E}_1 = \mathrm{Re}\boldsymbol{E}_1$ and $\mathbf{j}^{ac} = \mathrm{Re}\boldsymbol{J}$ are connected by the relation $\mathbf{E}_1 = \hat{\rho}^{ac}\mathbf{j}$, where the *ac*-response resistivity tensor is

$$\hat{\rho}^{ac} = \begin{pmatrix} \rho^{ac}_{xx} & \rho^{ac}_{xy} \\ \rho^{ac}_{yx} & \rho^{ac}_{yy} \end{pmatrix} = \mathrm{Re}\{\hat{Z}(\omega)e^{i\omega t}\}. \tag{15}$$

The longitudinal and transverse (with respect to the *ac* current direction which is the same, for simplicity, as the *dc* current direction) *ac* magnetoresistivities are

$$\rho^{ac}_{\parallel} = \mathrm{Re}\{Z_{\parallel}e^{i\omega t}\}, \quad \rho^{ac}_{\perp} = \mathrm{Re}\{Z_{\perp}e^{i\omega t}\}, \tag{16}$$

where the equations for $Z_{\parallel}$ and $Z_{\perp}$ are similar to Eqs. (9) with the only difference in a change of ν^{ω}_0 for Z_1 and $\rho_{\parallel,\perp}$ for $Z_{\parallel,\perp}$; note also that for $\alpha \neq 0, \pi/2$ even and odd components of Z_1 and $Z_{\parallel,\perp}$ should be considered.

In order to calculate the power absorbed per unit volume $\bar{\mathcal{P}}$ (and averaged over the period of an *ac* cycle) we use the standard relation $\bar{\mathcal{P}} = (1/2)\mathrm{Re}(\boldsymbol{E}_1 \cdot \boldsymbol{J})$ where $\boldsymbol{E}_1$ and $\boldsymbol{J}$ are the complex amplitudes of the *ac* electric field and current density, respectively. Then we can show that $\bar{\mathcal{P}} = (j^2/2)\bar{\rho} \equiv (j^2/2)\mathrm{Re}Z_{\parallel}$, where $\bar{\rho} = (\rho_f/D)[D\sin^2\alpha + (1 - D\sin^2\alpha)\mathrm{Re}Z_1]$.

In conclusion, the considered exactly solvable two-dimensional model of the vortex dynamics is of great interest since a very rich physics is expected from combination of a strong *dc* and *ac* driving, arbitrary value of the Hall effect, and the low temperature-mediated vortex hopping (or running) in a washboard pinning potential. Experimental realization of this model in thin-film geometry [5,6] and in layered HTSCs (which demonstrate WPPP in *c*-direction) opens up a possibility for a variety of experimental studies of directed motion of vortices under $(dc + ac)$-driving simply by measuring longitudinal and transverse voltages.

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